Exercise 7

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - 2y' + 5y = \sin x, \quad y(0) = 1, \quad y'(0) = 1$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' + 5y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - 2(re^{rx}) + 5(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 5 = 0$$

Solve for r.

$$r = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

$$r = \{1 - 2i, 1 + 2i\}$$

Two solutions to the ODE are $e^{(1-2i)x}$ and $e^{(1+2i)x}$. By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{(1-2i)x} + C_2 e^{(1+2i)x} \\ &= C_1 e^x e^{-2ix} + C_2 e^x e^{2ix} \\ &= e^x (C_1 e^{-2ix} + C_2 e^{2ix}) \\ &= e^x [C_1(\cos 2x - i \sin 2x) + C_2(\cos 2x + i \sin 2x)] \\ &= e^x [(C_1 + C_2)\cos 2x + (-iC_1 + iC_2)\sin 2x] \\ &= e^x (C_3 \cos 2x + C_4 \sin 2x). \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 2y_p' + 5y_p = \sin x \tag{2}$$

Since the inhomogeneous term is a sine function, the particular solution is $y_p = A\cos x + B\sin x$.

$$y_p = A\cos x + B\sin x \quad \rightarrow \quad y_p' = -A\sin x + B\cos x \quad \rightarrow \quad y_p'' = -A\cos x - B\sin x$$

Substitute these formulas into equation (2).

$$(-A\cos x - B\sin x) - 2(-A\sin x + B\cos x) + 5(A\cos x + B\sin x) = \sin x$$

$$(-A - 2B + 5A)\cos x + (-B + 2A + 5B)\sin x = \sin x$$

Match the coefficients on both sides to get a system of equations for A and B.

$$\left.\begin{aligned} -A - 2B + 5A &= 0 \\ -B + 2A + 5B &= 1 \end{aligned}\right\}$$

Solving it yields

$$A = \frac{1}{10} \quad \text{and} \quad B = \frac{1}{5},$$

which means the particular solution is

$$y_p = \frac{1}{10}\cos x + \frac{1}{5}\sin x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= e^x(C_3\cos 2x + C_4\sin 2x) + \frac{1}{10}\cos x + \frac{1}{5}\sin x, \end{aligned}$$

where C_3 and C_4 are arbitrary constants. Differentiate it with respect to x.

$$y'(x) = e^x(C_3\cos 2x + C_4\sin 2x) + e^x(-2C_3\sin 2x + 2C_4\cos 2x) - \frac{1}{10}\sin x + \frac{1}{5}\cos x$$

Apply the boundary conditions to determine C_3 and C_4.

$$y(0) = C_3 + \frac{1}{10} = 1$$

$$y'(0) = C_3 + 2C_4 + \frac{1}{5} = 1$$

Solving the system yields $C_3 = 9/10$ and $C_4 = -1/20$. Therefore,

$$y(x) = e^x\left(\frac{9}{10}\cos 2x - \frac{1}{20}\sin 2x\right) + \frac{1}{10}\cos x + \frac{1}{5}\sin x.$$